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Bianchi Type II, VIII, and IX Cosmological Models in Lyra's Geometry

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The Bianchi type II, VIII, and IX models are investigated in Lyra's geometry (in normal gauge) when the gauge function β is time dependent. The physical behavior of these models in vacuum and in the presence of the Zeldovich fluid is discussed.

1. INTRODUCTION

Weyl (1918) introduced a generalization of Riemannian geometry in an attempt to unify gravitation and electromagnetism. Lyra (1951) proposed another modification of Riemannian geometry in which, in contrast to Weyl's geometry, the connection is metric preserving and length transfer is integrable as in Riemannian geometry. Lyra introduced a gauge function into the structureless manifold as a result of which a displacement field arises naturally.

The Lyra geometry is more in keeping with the spirit of Einstein's principle of geometrization since both the scalar and tensor fields have more or less a geometrical significance. Furthermore, the present theory predicts the same effects, within observational limits, as far as the classical solar system tests are concerned as well as the tests based on the linearized form of the field equations (Halford, 1972) (see also Sen, 1960; Sen and Dunn, 1971; Sen and Vanstone, 1972). However, the energy-momentum tensor $T^{\mu\nu}$ is not conserved in Lyra's geometry.

Several authors (Sen, 1957; Halford, 1970; Beesham, 1968*a*,*b*; Bhamra, 1974; Kalyanshetti and Waghmode, 1982; Karade and Borikar, 1978;

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Reddy and Innaiah, 1985, 1986; Reddy and Venkateswarlu, 1987) have studied cosmology in Lyra's geometry with a constant displacement field, which plays the same role as the cosmological constant in the usual treatment. Sen (1957) constructed a static model with finite density similar to the static Einstein universe, but a significant difference was that the model exhibited a red shift. Halford (1970) studied Robertson–Walker models in Lyra's geometry. All these authors studied cosmological models in Lyra's geometry for a gauge function independent of time.

In cosmological models in Lyra's geometry the constant displacement field plays a role analogous to the cosmological constant in general relativity. Soleng (1987) has pointed out that the cosmology based on Lyra's geometry with constant gauge vector ϕ_μ will either include a creation field and be identical to Hoyle's creation field cosmology (Hoyle, 1948; Hoyle and Narlikar, 1963) or contain a special vacuum field which together with the guage term may be considered as a cosmological term. In the latter case the solutions are identical to the general relativistic cosmologies with a cosmological term.

In this work we consider Bianchi type II, VIII, and IX models in the case when the displacement field is time-dependent. Similar results can be obtained in Hoyle's creation field theory (Hoyle, 1948) if the creation field is assumed to be time-dependent. Such investigations have not been undertaken in Hoyle's theory so far. Singh and Singh (1990, 1991*a*,*b*) investigated Bianchi type I, III, V, and VI_0 and Kantowski–Sachs cosmological models in the Lyra manifold. Beesham (1988) investigated FRW cosmological models in the Lyra manifold with a time-dependent displacement field. Singh and Singh (1991*c*) discussed FRW models in the Lyra manifold with constant deceleration parameter which are relevant to the study of inflationary cosmology.

2. FIELD EQUATIONS

The field equations are (Sen, 1957)

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+\tfrac{3}{2}\phi_\mu\phi_\nu-\tfrac{3}{4}g_{\mu\nu}\phi_\alpha\phi^\alpha=-\chi T_{\mu\nu} \tag{2.1}$$

where ϕ_μ is a displacement field vector defined as

$$\phi_\mu=(0,0,0,\beta) \tag{2.2}$$

with $\beta=\beta(t)$. The other symbols have their usual meaning as in Riemannian geometry. We take the energy-momentum tensor $T_{\mu\nu}$ for the perfect fluid as

$$T_{\mu\nu}=(\rho+p)u_\mu u_\nu-pg_{\mu\nu} \tag{2.3}$$

together with $u^i=[0, 0, 0, (g_{44})^{-1/2}]$. Here p and ρ are pressure and density, respectively. The equation of state for the fluid is taken as

$$p=(\lambda-1)\rho, \qquad 1\le\lambda\le 2 \tag{2.4}$$

3. BIANCHI TYPE II MODEL

The Bianchi type II metric is given by

$$dS^2=dt^2-S^2\,dx^2-R^2\,dy^2-(R^2y^2+\tfrac{1}{4}S^2y^4)\,dz^2-S^2y^2\,dx\,dz \tag{3.1}$$

where

$$S=S(t), \qquad R=R(t)$$

For the metric (3.1) the field equations (2.1)–(2.4) lead to

$$2\frac{\ddot{R}}{R}+\left(\frac{\dot{R}}{R}\right)^2-\frac{3}{4}\frac{S^2}{R^4}=-\chi p-\frac{3}{4}\beta^2 \tag{3.2}$$

$$\frac{\ddot{R}}{R}+\frac{\ddot{S}}{S}+\frac{\dot{R}\dot{S}}{RS}+\frac{1}{4}\frac{S^2}{R^4}=-\chi p-\frac{3}{4}\beta^2 \tag{3.3}$$

$$2\frac{\dot{R}\dot{S}}{RS}+\left(\frac{\dot{R}}{R}\right)^2-\frac{1}{4}\frac{S^2}{R^4}=\chi\rho+\frac{3}{4}\beta^2 \tag{3.4}$$

The energy conservation equation is

$$\chi\dot{\rho}+\frac{3}{2}\beta\dot{\beta}+\left[\chi(p+\rho)+\frac{3}{2}\beta^2\right]\left(\frac{\dot{S}}{S}+2\frac{\dot{R}}{R}\right)=0 \tag{3.5}$$

An overdot denotes differentiation with respect to t. Making the transformation of the time coordinate by

$$dt=R^2S\,d\eta \tag{3.6}$$

we find that equations (3.2)–(3.5) reduce to

$$2\frac{R''}{R}-3\left(\frac{R'}{R}\right)^2-2\frac{R'S'}{RS}-\frac{3}{4}S^4=-\left(\chi p+\frac{3}{4}\beta^2\right)R^4S^2 \tag{3.2a}$$

$$\frac{R''}{R}-2\left(\frac{R'}{R}\right)^2-2\frac{R'S'}{RS}+\frac{S''}{S}-\left(\frac{S'}{S}\right)^2+\frac{1}{4}S^4$$
$$=-(\chi p+\tfrac{3}{4}\beta^2)R^4S^2 \tag{3.3a}$$

$$2\frac{R'S'}{RS}+\left(\frac{R'}{R}\right)^2-\frac{1}{4}S^4=\left(x\rho+\frac{3}{4}\beta^2\right)R^4S^2 \tag{3.4a}$$

and the energy conservation equation is

$$x\rho' + \frac{3}{2}\beta\beta' + \left[\chi(p+\rho) + \frac{3}{2}\beta^2\right]\left(\frac{S'}{S} + 2\frac{R'}{R}\right) = 0 \tag{3.5a}$$

From (3.2a)–(3.4a) we have

$$2\frac{R''}{R} - 2\left(\frac{R'}{R}\right)^2 - S^4 = \chi(\rho - p)R^4S^2 \tag{3.7}$$

$$\frac{R''}{R} - \left(\frac{R'}{R}\right)^2 + \frac{S''}{S} - \left(\frac{S'}{S}\right)^2 = \chi(\rho - p)R^4S^2 \tag{3.8}$$

Primes denote differentiation with respect to η.

Finally, we have only three equations (3.4a), (3.7), and (3.8) in five unknowns, R, S, β, ρ, and p.

It is difficult to solve the field equations in general, and therefore we consider some physically interesting cases.

Case I. Empty Universive ($p = \rho = 0$)

In this case equations (3.4a), (3.7), (3.8), and (3.5a) reduce to

$$\frac{R''}{R} - \left(\frac{R'}{R}\right)^2 - \frac{1}{2}S^4 = 0 \tag{3.9}$$

$$\frac{R''}{R} - \left(\frac{R'}{R}\right)^2 + \frac{S''}{S} - \left(\frac{S'}{S}\right)^2 = 0 \tag{3.10}$$

$$2\frac{R'S'}{RS} + \left(\frac{R'}{R}\right)^2 - \frac{1}{4}S^4 = \frac{3}{4}R^4S^2 \tag{3.11}$$

$$\frac{3}{2}\beta\beta' + \frac{3}{2}\beta^2\left(\frac{S'}{S} + 2\frac{R'}{R}\right) = 0 \tag{3.12}$$

From (3.9) and (3.10), we get

$$\frac{S''}{S} - \left(\frac{S'}{S}\right)^2 + \frac{1}{2}S^4 = 0 \tag{3.13}$$

Solving equations (3.9), (3.10), and (3.13), we obtain

$$\left.\begin{aligned} S^2 &= c_3 \operatorname{sech}(c_3\eta + c_4) \\ RS &= e^{c_1\eta + c_2} \\ R^2 &= (1/C_3)\cosh(c_3\eta + c_4)\, e^{2(c_1\eta + c_2)} \end{aligned}\right\} \tag{3.14}$$

Here c_1, c_2, c_3, c_4 are arbitrary constants.

Using (3.14) in (3.11), we get

$$\beta(t) = \left[\frac{c_3}{3}(4c_1^2 - c_3^2)\right]^{1/2} \frac{e^{-2(c_1\eta + c_2)}}{[\cosh(c_3\eta + c_4)]^{1/2}} \tag{3.15}$$

Using (3.14) and (3.15) in (3.12), we see that the energy conservation equation is identically satisfied.

The gauge function $\beta(t)$ is real or imaginary according as:

(i) $c_3 > 0$, $2|c_1| > |c_3|$ or (ii) $c_3 > 0$, $2|c_1| < |c_3|$

and

(i) $c_3 < 0$, $2|c_1| < |c_3|$ or (ii) $c_3 < 0$, $2|c_1| > |c_3|$

Physical Behavior of the Model

From equation (3.15), we see that

$$\beta(t) \propto \frac{e^{-2(c_1\eta + c_2)}}{[\cosh(c_3\eta + c_4)]^{1/2}} \tag{3.16}$$

The Ricci scalar is

$$R = 2c_3\left(\frac{c_3^2}{4} - c_1^2\right)\frac{e^{-4(c_1\eta + c_2)}}{\cosh(c_3\eta + c_4)} \tag{3.17}$$

The dynamical parameters are as follows (Raychaudhuri, 1955).

The shear:

$$\begin{aligned} \sigma^2 &= \frac{1}{12}\left[\left(\frac{\dot g_{11}}{g_{11}} - \frac{\dot g_{22}}{g_{22}}\right)^2 + \left(\frac{\dot g_{22}}{g_{22}} - \frac{\dot g_{33}}{g_{33}}\right)^2 + \left(\frac{\dot g_{33}}{g_{33}} - \frac{\dot g_{11}}{g_{11}}\right)^2\right] \\ &\quad + \tfrac{1}{2}(g^{11}g^{22}\dot g_{12}^2 + g^{11}g^{33}\dot g_{13}^2 + g^{22}g^{33}\dot g_{23}^2) \\ \sigma^2 &= \frac{2}{3}\left(\frac{\dot R}{R} - \frac{\dot S}{S}\right)^2 \\ \sigma^2 &= \frac{2c_3}{3} e^{-4(c_1\eta + c_2)} \operatorname{sech}(c_3\eta + c_4) \\ &\quad \times [c_1 + c_3 \tanh(c_3\eta + c_4)]^2 \end{aligned} \tag{3.18}$$

Scalar of expansion:

$$\theta = u^k{}_{;k} = \frac{\dot S}{S} + 2\frac{\dot R}{R} = 3\frac{\dot V}{V}$$
$$\theta = \left[2c_1 + \frac{c_3}{2}\tanh(c_3\eta + c_4)\right] \tag{3.19}$$

Hubble parameter:

$$H = \frac{\dot V}{V} = \frac{1}{3}\theta$$

Deceleration parameter:

$$q = -\frac{V\ddot V}{\dot V^2}$$
$$q = -1 - \frac{3}{2}c_3^2\,\mathrm{sech}^2(c_3\eta + c_4)\left[2c_1 + \frac{c_3}{2}\tanh(c_3\eta + c_4)\right]^{-2} \tag{3.20}$$

Rotation tensor:

$$\omega_{\mu\nu} = \tfrac{1}{2}[g_{\mu 4,\nu} - g_{\nu 4,\mu}] = 0$$
$$\frac{\sigma^2}{\theta} = \frac{2c_3}{3}e^{-4(c_1\eta + c_2)}\,\mathrm{sech}(c_3\eta + c_4) \times \frac{[c_1 + c_3\tanh(c_3\eta + c_4)]^2}{2c_1 + \frac{1}{2}c_3\tanh(c_3\eta + c_4)} \tag{3.21}$$

Spatial volume:

$$V^3 = \sqrt{-g} \propto SR^2 \tag{3.22}$$

Case II. Zeldovich Fluid $p = \rho$

In this case equations (3.4a), (3.5a), (3.7), and (3.8) reduce to

$$\frac{R''}{R} - \left(\frac{R'}{R}\right)^2 - \frac{1}{2}S^4 = 0 \tag{3.23}$$

$$\frac{R''}{R} - \left(\frac{R'}{R}\right)^2 + \frac{S''}{S} - \left(\frac{S'}{S}\right)^2 = 0 \tag{3.24}$$

$$2\frac{R'S'}{RS} + \left(\frac{R'}{R}\right)^2 - \frac{1}{4}S^4 = \left(\chi\rho + \frac{3}{4}\beta^2\right)R^4S^2 \tag{3.25}$$

The energy conservation equation is

$$\chi\rho'+\frac{3}{2}\beta\beta'+\left(2\chi\rho+\frac{3}{2}\beta^2\right)\left(\frac{S'}{S}+2\frac{R'}{R}\right)=0 \tag{3.26}$$

From (3.23) and (3.24), we get

$$\frac{S''}{S}-\left(\frac{S'}{S}\right)^2+\frac{1}{2}S^4=0 \tag{3.27}$$

Solving equations (3.23), (3.24), and (3.27), we get

$$\left.\begin{aligned} &S^2=c_3\,\mathrm{sech}(c_3\eta+c_4)\\ &RS=e^{c_1\eta+c_2}\\ &R^2=\frac{1}{c_3}\cosh(c_3\eta+c_4)\,e^{2(c_1\eta+c_2)} \end{aligned}\right\} \tag{3.28}$$

Using (3.28) in (3.25), we get

$$p=\rho=-\frac{3}{4\chi}\beta^2+\frac{c_3}{\chi}\left(c_1^2-\frac{c_3^2}{4}\right)\frac{e^{-4(c_1\eta+c_2)}}{\cosh(c_3\eta+c_4)} \tag{3.29}$$

Here β remains an arbitrary function of t.

Using (3.28) and (3.29) in (3.26), we find that the energy conservation equation is identically satisfied.

Physical Behavior of the Model

The Ricci scalar is

$$R=2c_3\left(\frac{c_3^2}{4}-c_1^2\right)\frac{e^{-4(c_1\eta+c_2)}}{\cosh(c_3\eta+c_4)} \tag{3.30}$$

The dynamical parameters are as follows (Ellis, 1971).

Shear tensor:

$$\sigma_{ij}=\tfrac{1}{2}(u_{i;j}+u_{j;i})+\tfrac{1}{2}(\dot{u}_iu_j+\dot{u}_ju_i)-\tfrac{1}{3}h_{ij}(u^{k}{}_{;k})$$

where

$$\sigma^2=\tfrac{1}{2}\sigma_{ij}\sigma^{ij},\qquad h_{ij}=g_{ij}-u_iu_j$$

Shear:

$$\sigma^2=\frac{2}{3}\left(\frac{\dot{R}}{R}-\frac{\dot{S}}{S}\right)^2$$
$$\sigma^2=\frac{2c_3}{3}e^{-4(c_1\eta+c_2)}\operatorname{sech}(c_3\eta+c_4)[c_1+c_3\tanh(c_3\eta+c_4)]^2 \tag{3.31}$$

Scalar of expansion:

$$\theta=u^k{}_{;k}$$
$$\theta=\left(\frac{\dot{S}}{S}+2\frac{\dot{R}}{R}\right)=2c_1+\frac{c_3}{2}\tanh(c_3\eta+c_4)$$

Rotation tensor:

$$\omega_{ij}=\tfrac{1}{2}(u_{i;j}-u_{j;i})-\tfrac{1}{2}(\dot{u}_iu_j-\dot{u}_ju_i)$$

Rotation:

$$\omega^2=\tfrac{1}{2}\omega_{ij}\omega^{ij}$$

Here $\omega_{ij}=0$.

Hubble parameter:

$$H=\tfrac{1}{3}\theta$$

Deceleration parameter:

$$q=-\frac{V\ddot{V}}{\dot{V}^2}$$
$$q=-1-\frac{3}{2}c_3^2\operatorname{sech}^2(c_3\eta+c_4)\left[2c_1+\frac{c_3}{2}\tanh(c_3\eta+c_4)\right]^{-2} \tag{3.32}$$

$$\frac{\sigma^2}{\theta}=\frac{2c_3}{3}e^{-4(c_1\eta+c_2)}\operatorname{sech}(c_3\eta+c_4)\times\frac{[c_1+c_3\tanh(c_3\eta+c_4)]^2}{2c_1+\frac{1}{2}c_3\tanh(c_3\eta+c_4)} \tag{3.33}$$

The relative anisotropy:

$$\frac{\sigma^2}{\rho}=\frac{2c_3}{3}e^{-4(c_1\eta+c_2)}\operatorname{sech}(c_3\eta+c_4)[c_1+c_3\tanh(c_3\eta+c_4)]^2$$
$$\times\left[-\frac{3}{4\chi}\beta^2+\frac{c_3}{\chi}\left(c_1^2-\frac{c_3^2}{4}\right)\frac{e^{-4(c_1\eta+c_2)}}{\cosh(c_3\eta+c_4)}\right]^{-1} \tag{3.34}$$

4. BIANCHI TYPE VIII MODEL

The Bianchi type VIII metric is given by

$$dS^2 = dt^2 - S^2\,dx^2 - R^2\,dy^2 - (R^2\sinh^2 y + S^2\cosh^2 y)\,dz^2 - 2S^2\cosh y\,dx\,dz \tag{4.1}$$

where

$$R = R(t), \qquad S = S(t)$$

The field equations (2.1)–(2.4) for the metric (4.1) lead to

$$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 - \frac{1}{R^2} - \frac{3}{4}\frac{S^2}{R^4} = -\chi p - \frac{3}{4}\beta^2 \tag{4.2}$$

$$\frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} + \frac{\dot{R}\dot{S}}{RS} + \frac{1}{4}\frac{S^2}{R^4} = -\chi p - \frac{3}{4}\beta^2 \tag{4.3}$$

$$2\frac{\dot{R}\dot{S}}{RS} + \left(\frac{\dot{R}}{R}\right)^2 - \frac{1}{R^2} - \frac{1}{4}\frac{S^2}{R^4} = \chi\rho + \frac{3}{4}\beta^2 \tag{4.4}$$

The energy conservation equation is

$$\chi\dot{\rho} + \frac{3}{2}\beta\dot{\beta} + \left[\chi(\rho + p) + \frac{3}{2}\beta^2\right]\left(\frac{\dot{S}}{S} + 2\frac{\dot{R}}{R}\right) = 0 \tag{4.5}$$

Here the dot denotes differentiation with respect to t.

Using the transformation (3.6), we find that equations (4.2)–(4.5) reduce to

$$2\frac{R''}{R} - 3\left(\frac{R'}{R}\right)^2 - 2\frac{R'S'}{RS} - R^2S^2 - \frac{3}{4}\frac{S^2}{R^4} = -(\chi p + \tfrac{3}{4}\beta^2)R^4S^2 \tag{4.2a}$$

$$\frac{R''}{R} - 2\left(\frac{R'}{R}\right)^2 - 2\frac{R'S'}{RS} + \frac{S''}{S} - \left(\frac{S'}{S}\right)^2 + \frac{1}{4}S^4 = -(\chi p + \tfrac{3}{4}\beta^2)R^4S^2 \tag{4.3a}$$

$$2\frac{R'S'}{RS} + \left(\frac{R'}{R}\right)^2 - R^2S^2 - \frac{1}{4}S^4 = \left(\chi\rho + \frac{3}{4}\beta^2\right)R^4S^2 \tag{4.4a}$$

$$\chi\rho' + \frac{3}{2}\beta\beta' + \left[\chi(\rho + p) + \frac{3}{2}\beta^2\right]\left(\frac{S'}{S} + 2\frac{R'}{R}\right) = 0 \tag{4.5a}$$

Here primes denote differentiation with respect to η.

From (4.2a)–(4.4a) we obtain

$$2\frac{R''}{R}-2\left(\frac{R'}{R}\right)^2-2R^2S^2-S^4=\chi(\rho-p)R^4S^2 \tag{4.6}$$

$$\frac{R''}{R}-\left(\frac{R'}{R}\right)^2+\frac{S''}{S}-\left(\frac{S'}{S}\right)^2-R^2S^2=\chi(\rho-p)R^4S^2 \tag{4.7}$$

It is difficult to find a general solution and therefore we consider some particular cases.

Case I. Empty Universe ($p=\rho=0$)

In this case equations (4.4a), (4.6), and (4.7) reduce to

$$\frac{R''}{R}-\left(\frac{R'}{R}\right)^2-R^2S^2=\frac{1}{2}S^4 \tag{4.8}$$

$$\frac{R''}{R}-\left(\frac{R'}{R}\right)^2+\frac{S''}{S}-\left(\frac{S'}{S}\right)^2-R^2S^2=0 \tag{4.9}$$

$$2\frac{R'S'}{RS}+\left(\frac{R'}{R}\right)^2-R^2S^2-\frac{1}{4}S^4=\frac{3}{4}\beta^2R^4S^4 \tag{4.10}$$

The energy conservation equation is

$$\frac{3}{2}\beta\beta'+\frac{3}{2}\beta^2\left(\frac{S'}{S}+2\frac{R'}{R}\right)=0 \tag{4.11}$$

From (4.8) and (4.9), we get

$$\frac{S''}{S}-\left(\frac{S'}{S}\right)^2+\frac{1}{2}S^4=0 \tag{4.12}$$

Solving equations (4.8), (4.9), and (4.12), we easily obtain

$$\left.\begin{aligned} S^2&=c_3\,\mathrm{sech}(c_3\eta+c_4)\\ RS&=c_1\,\mathrm{cosech}(c_1\eta+c_2)\\ R^2&=\frac{c_1^2\,\mathrm{cosech}^2(c_1\eta+c_2)}{c_3\,\mathrm{sech}(c_3\eta+c_4)}\end{aligned}\right\} \tag{4.13}$$

Here c_1, c_2, c_3, c_4 are arbitrarily constants.

Using (4.13) in (4.10), we get

$$\beta(t)=\left[\frac{c_3}{3c_1^4}(4c_1^2-c_3^2)\right]^{1/2}\frac{[\operatorname{sech}(c_3\eta+c_4)]^{1/2}}{\operatorname{cosech}^2(c_1\eta+c_2)} \tag{4.14}$$

Using (4.13) and (4.14) in (4.11), we see that the energy conservation equation is identically satisfied.

The gauge function $\beta(t)$ is real or imaginary according as:

(i) $c_3>0,\ 2|c_1|>|c_3|$ or (ii) $c_3>0,\ 2|c_1|<|c_3|$

and

(i) $c_3<0,\ 2|c_1|<|c_3|$ or (ii) $c_3<0,\ 2|c_1|>|c_3|$

The Physical Behavior of the Model

The Ricci scalar is

$$R=2\frac{c_3}{c_1^4}\left(\frac{c_3^2}{4}-c_1^2\right)\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{cosech}^4(c_1\eta+c_2)} \tag{4.15}$$

The dynamical parameters are as follows.

Shear scalar:

$$\sigma^2=\frac{2c_3}{3c_1^4}\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{cosech}^4(c_1\eta+c_2)} \times[c_3\tanh(c_3\eta+c_4)-c_1\coth(c_1\eta+c_2)]^2 \tag{4.16}$$

Scalar of expansion:

$$\theta=\frac{c_3}{2}\tanh(c_3\eta+c_4)-2c_1\coth(c_1\eta+c_2)$$

Hubble parameter:

$$H=\tfrac{1}{3}\theta$$

Deceleration parameter:

$$q=-\frac{V\ddot{V}}{\dot{V}^2}$$

$$q=-1-3\frac{2c_1^2\operatorname{cosech}^2(c_1\eta+c_2)+\frac{1}{2}c_3^2\operatorname{sech}^2(c_3\eta+c_4)}{\frac{1}{2}c_3\tanh(c_3\eta+c_4)-2c_1\coth(c_1\eta+c_2)}$$

The rotation tensor:

$$\omega_{ij}=0$$

$$\frac{\sigma^2}{\theta}=\frac{2c_3}{3c_1^4}\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{cosech}^4(c_1\eta+c_2)} \times\frac{[c_3\tanh(c_3\eta+c_4)-c_1\coth(c_1\eta+c_2)]^2}{\frac{1}{2}c_3\tanh(c_3\eta+c_4)-2c_1\coth(c_1\eta+c_2)} \tag{4.17}$$

Case II. Zeldovich Fluid ($p=\rho$)

In this case equations (4.4a), (4.6), and (4.7) reduce to

$$\frac{R''}{R}-\left(\frac{R'}{R}\right)^2-R^2S^2=\frac{1}{2}S^4 \tag{4.18}$$

$$\frac{R''}{R}-\left(\frac{R'}{R}\right)^2+\frac{S''}{S}-\left(\frac{S'}{S}\right)^2-R^2S^2=0 \tag{4.19}$$

$$2\frac{R'S'}{RS}+\left(\frac{R'}{R}\right)^2-R^2S^2-\frac{1}{4}S^4=\left(\chi\rho+\frac{3}{4}\beta^2\right)R^4S^2 \tag{4.20}$$

and the energy conservation equation is

$$\chi\rho'+\frac{3}{2}\beta\beta'+\left(2\chi\rho+\frac{3}{2}\beta^2\right)\left(\frac{S'}{S}+2\frac{R'}{R}\right)=0 \tag{4.21}$$

From (4.18) and (4.19), we get

$$\frac{S''}{S}-\left(\frac{S'}{S}\right)^2+\frac{1}{2}S^4=0 \tag{4.22}$$

Solving equations (4.18), (4.19), and (4.22), we obtain

$$\begin{aligned} S^2&=c_3\operatorname{sech}(c_3\eta+c_4)\\ RS&=c_1\operatorname{cosech}(c_1\eta+c_2)\\ R^2&=\frac{c_1^2\operatorname{cosech}^2(c_1\eta+c_2)}{c_3\operatorname{sech}(c_3\eta+c_4)}\end{aligned} \tag{4.23}$$

Using (4.23) in (4.20), we get

$$p=\rho=-\frac{3}{4\chi}\beta^2+\frac{c_3}{\chi c_1^4}\left(c_1^2-\frac{c_3^2}{4}\right)\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{cosech}^4(c_1\eta+c_2)} \tag{4.24}$$

Here β remains an arbitrary function of t.

Using (4.23) and (4.24) in (4.21), we see that the energy conservation equation is identically satisfied.

Physical Behavior of the Model

The Ricci scalar is

$$R=\frac{2c_3}{c_1^4}\left(\frac{c_3^2}{4}-c_1^2\right)\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{cosech}^4(c_1\eta+c_2)} \tag{4.25}$$

The dynamical parameters are as follows.

Shear scalar:

$$\sigma^2=\frac{2c_3}{3c_1^4}\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{cosech}^4(c_1\eta+c_2)} \times[c_3\tanh(c_3\eta+c_4)-c_1\coth(c_1\eta+c_2)]^2 \tag{4.26}$$

Scalar of expansion:

$$\theta=\left[\frac{c_3}{2}\tanh(c_3\eta+c_4)-2c_1\coth(c_1\eta+c_2)\right]$$

Hubble parameter:

$$H=\tfrac{1}{3}\theta$$

Deceleration parameter:

$$q=-1-3\,\frac{2c_1^2\operatorname{cosech}^2(c_1\eta+c_2)+\frac{1}{2}c_3^2\operatorname{sech}^2(c_3\eta+c_4)}{\frac{1}{2}c_3\tanh(c_3\eta+c_4)-2c_1\coth(c_1\eta+c_2)}$$

Rotation tensor:

$$\omega_{ij}=0$$

$$\frac{\sigma^2}{\theta}=\frac{2c_3}{3c_1^4}\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{cosech}^4(c_1\eta+c_2)} \times\frac{[c_3\tanh(c_3\eta+c_4)-c_1\coth(c_1\eta+c_2)]^2}{\frac{1}{2}c_3\tanh(c_3\eta+c_4)-2c_1\coth(c_1\eta+c_2)} \tag{4.27}$$

The relative anisotropy:

$$\frac{\sigma^2}{\rho}=\frac{2c_3}{3c_1^4}\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{cosech}^4(c_1\eta+c_2)}$$
$$\times[c_3\tanh(c_3\eta+c_4)-c_1\coth(c_1\eta+c_2)]^2$$
$$\times\left\{-\frac{3}{4\chi}\beta^2+\frac{c_3}{c_1^4\chi}\left(c_1^2-\frac{c_3^2}{4}\right)\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{cosech}^4(c_1\eta+c_2)}\right\}^{-1} \tag{4.28}$$

5. BIANCHI TYPE IX MODEL

The Bianchi type IX metric is given by

$$dS^2=dt^2-S^2\,dx^2-R^2\,dy^2-(R^2\sin^2 y+S^2\cos^2 y)\,dz^2$$
$$+2S^2\cos y\,dx\,dz \tag{5.1}$$

where

$$R=R(t),\qquad S=S(t)$$

The field equations (2.1)–(2.4) for the metric (5.1) lead to

$$2\frac{\ddot R}{R}+\left(\frac{\dot R}{R}\right)^2+\frac{1}{R^2}-\frac{3}{4}\frac{S^2}{R^4}=-\chi p-\frac{3}{4}\beta^2 \tag{5.2}$$

$$\frac{\ddot R}{R}+\frac{\ddot S}{S}+\frac{\dot R\dot S}{RS}+\frac{1}{4}\frac{S^2}{R^4}=-\chi p-\frac{3}{4}\beta^2 \tag{5.3}$$

$$2\frac{\dot R\dot S}{RS}+\left(\frac{\dot R}{R}\right)^2+\frac{1}{R^2}-\frac{1}{4}\frac{S^2}{R^4}=\chi\rho+\frac{3}{4}\beta^2 \tag{5.4}$$

The energy conservation equation is

$$\chi\dot\rho+\frac{3}{2}\beta\dot\beta+\left[\chi(p+\rho)+\frac{3}{2}\beta^2\right]\left(\frac{\dot S}{S}+2\frac{\dot R}{R}\right)=0 \tag{5.5}$$

Here the dot denotes differentiation with respect to t.

Using the transformation (3.6), we find that equations (5.2)–(5.5) reduce to

$$2\frac{R''}{R}-3\left(\frac{R'}{R}\right)^2-2\frac{R'S'}{RS}+R^2S^2-\frac{3}{4}\frac{S^2}{R^4}$$
$$=-\left(\chi p+\frac{3}{4}\beta^2\right)R^4S^2 \tag{5.2a}$$

$$\frac{R''}{R}-2\left(\frac{R'}{R}\right)^2-2\frac{R'S'}{RS}+\frac{S''}{S}-\left(\frac{S'}{S}\right)^2+\frac{1}{4}S^4$$
$$=-(\chi p+\tfrac{3}{4}\beta^2)R^4S^2 \tag{5.3a}$$

$$2\frac{R'S'}{RS}+\left(\frac{R'}{R}\right)^2+R^2S^2-\frac{1}{4}S^4=\left(\chi\rho+\frac{3}{4}\beta^2\right)R^4S^2 \tag{5.4a}$$

$$\chi\rho'+\frac{3}{2}\beta\beta'+\left[\chi(\rho+p)+\frac{3}{2}\beta^2\right]\left(\frac{S'}{S}+2\frac{R'}{R}\right)=0 \tag{5.5a}$$

From (5.2a)–(5.4a) we easily obtain

$$2\frac{R''}{R}-2\left(\frac{R'}{R}\right)^2+2R^2S^2-S^4=\chi(\rho-p)R^4S^2 \tag{5.6}$$

$$\frac{R''}{R}-\left(\frac{R'}{R}\right)^2+\frac{S''}{S}-\left(\frac{S'}{S}\right)^2+R^2S^2=\chi(\rho-p)R^4S^2 \tag{5.7}$$

Here a prime denotes differentiation with respect to η.

It is difficult to find a general solution, so we consider some particular cases.

Case I. Empty Universe ($p=\rho=0$)

In this case equations (5.4a), (5.6), and (5.7) reduce to

$$\frac{R''}{R}-\left(\frac{R'}{R}\right)^2+R^2S^2=\frac{1}{2}S^4 \tag{5.8}$$

$$\frac{R''}{R}-\left(\frac{R'}{R}\right)^2+\frac{S''}{S}-\left(\frac{S'}{S}\right)^2+R^2S^2=0 \tag{5.9}$$

$$2\frac{R'S'}{RS}+\left(\frac{R'}{R}\right)^2+R^2S^2-\frac{1}{4}S^4=\frac{3}{4}\beta^2R^4S^2 \tag{5.10}$$

and the conservation equation is

$$\frac{3}{2}\beta\beta'+\frac{3}{2}\beta^2\left(\frac{S'}{S}+2\frac{R'}{R}\right)=0 \tag{5.11}$$

From (5.8) and (5.9), we get

$$\frac{S''}{S}-\left(\frac{S'}{S}\right)^2+\frac{1}{2}S^4=0 \tag{5.12}$$

Solving equations (5.8), (5.9), and (5.12), we get

$$\left.\begin{aligned} S^2&=c_3\,\mathrm{sech}(c_3\eta+c_4)\\ RS&=c_1\,\mathrm{sech}(c_1\eta+c_2)\\ R^2&=\frac{c_1^2\,\mathrm{sech}^2(c_1\eta+c_2)}{c_3\,\mathrm{sech}(c_3\eta+c_4)}\end{aligned}\right\} \tag{5.13}$$

Using (5.13) in (5.10), we get

$$\beta(t)=\frac{1}{c_1^2}\left[\frac{c_3}{3}(4c_1^2-c_3^2)\right]^{1/2}\frac{[\mathrm{sech}(c_3\eta+c_4)]^{1/2}}{\mathrm{sech}^2(c_1\eta+c_2)} \tag{5.14}$$

Using (5.13) and (5.14) in (5.11), we find that the energy conservation equation is identically satisfied.

The Physical Behavior of the Model

The Ricci scalar is

$$R=\frac{2}{c_1^4}c_3\left(\frac{c_3^2}{4}-c_1^2\right)\frac{\mathrm{sech}(c_3\eta+c_4)}{\mathrm{sech}^4(c_1\eta+c_2)} \tag{5.15}$$

The dynamical parameters are as follows.

Shear scalar:

$$\sigma^2=\frac{2c_3}{3c_1^4}\frac{\mathrm{sech}(c_3\eta+c_4)}{\mathrm{sech}^4(c_1\eta+c_2)} \times[c_3\tanh(c_3\eta+c_4)-c_1\tanh(c_1\eta+c_2)]^2 \tag{5.16}$$

Scalar of expansion:

$$\theta=\frac{c_3}{2}\tanh(c_3\eta+c_4)-2c_1\tanh(c_1\eta+c_2)$$

Hubble parameter:

$$H=\tfrac{1}{3}\theta$$

Deceleration parameter:

$$q=-1-3\,\frac{\frac{1}{2}c_3^2\,\mathrm{sech}^2(c_3\eta+c_4)-2c_1^2\,\mathrm{sech}^2(c_1\eta+c_2)}{\frac{1}{2}c_3\tanh(c_3\eta+c_4)-2c_1\tanh(c_1\eta+c_2)}$$

Rotation tensor:

$$\omega_{ij}=0$$

$$\frac{\sigma^2}{\theta}=\frac{2c_3}{3c_1^4}\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{sech}^4(c_1\eta+c_2)}$$
$$\times\frac{[c_3\tanh(c_3\eta+c_4)-c_1\tanh(c_1\eta+c_2)]^2}{\frac{1}{2}c_3\tanh(c_3\eta+c_4)-2c_1\tanh(c_1\eta+c_2)} \tag{5.17}$$

Case II. Zeldovich Fluid ($p=\rho$)

In this case equations (5.4a), (5.6), and (5.7) reduce to

$$\frac{R''}{R}-\left(\frac{R'}{R}\right)^2+R^2S^2=\frac{1}{2}S^4 \tag{5.18}$$

$$\frac{R''}{R}-\left(\frac{R'}{R}\right)^2+\frac{S''}{S}-\left(\frac{S'}{S}\right)^2+R^2S^2=0 \tag{5.19}$$

$$2\frac{R'S'}{RS}+\left(\frac{R'}{R}\right)^2+R^2S^2-\frac{1}{4}S^4=\left(\chi\rho+\frac{3}{4}\beta^2\right)R^4S^2 \tag{5.20}$$

and the conservation equation is

$$\chi\rho'+\frac{3}{2}\beta\beta'+\left(2\chi\rho+\frac{3}{2}\beta^2\right)\left(\frac{S'}{S}+2\frac{R'}{R}\right)=0 \tag{5.21}$$

From (5.18) and (5.19), we get

$$\frac{S''}{S}-\left(\frac{S'}{S}\right)^2+\frac{1}{2}S^4=0 \tag{5.22}$$

Solving equations (5.18), (5.19), and (5.22), we easily obtain

$$\left.\begin{aligned} S^2&=c_3\operatorname{sech}(c_3\eta+c_4)\\ RS&=c_1\operatorname{sech}(c_1\eta+c_2)\\ R^2&=\frac{c_1^2\operatorname{sech}^2(c_1\eta+c_2)}{c_3\operatorname{sech}(c_3\eta+c_4)}\end{aligned}\right\} \tag{5.23}$$

Using (5.23) in (5.20), we get

$$p=\rho=-\frac{3}{4\chi}\beta^2+\frac{c_3}{\chi c_1^4}\left(c_1^2-\frac{c_3^2}{4}\right)\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{sech}^4(c_1\eta+c_2)} \tag{5.24}$$

Here β remains an arbitrary function of t.

Using (5.23) and (5.24) in (5.21), we find that the energy conservarion equation is identically satisfied.

The Physical Behavior of the Model

The Ricci scalar is

$$R=\frac{2c_3}{c_1^4}\left(\frac{c_3^2}{4}-c_1^2\right)\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{sech}^4(c_1\eta+c_2)} \tag{5.25}$$

The dynamical parameters are as follows.

Shear scalar:

$$\begin{aligned}\sigma^2=&\frac{2c_3}{3c_1^4}\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{sech}^4(c_1\eta+c_2)}\\ &\times[c_3\tanh(c_3\eta+c_4)-c_1\tanh(c_1\eta+c_2)]\end{aligned} \tag{5.26}$$

Scalar of expansion:

$$\theta=\frac{c_3}{2}\tanh(c_3\eta+c_4)-2c_1\tanh(c_1\eta+c_2)$$

Hubble parameter:

$$H=\tfrac{1}{3}\theta$$

Deceleration parameter:

$$q=-1-3\frac{\frac{1}{2}c_3^2\operatorname{sech}^2(c_3\eta+c_4)-2c_1^2\operatorname{sech}^2(c_1\eta+c_2)}{\frac{1}{2}c_3\tanh(c_3\eta+c_4)-2c_1\tanh(c_1\eta+c_2)}$$

Rotation tensor:

$$\omega_{ij}=0$$

$$\begin{aligned}\frac{\sigma^2}{\theta}=&\frac{2c_3}{2c_1^4}\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{sech}^4(c_1\eta+c_2)}\\ &\times\frac{[c_3\tanh(c_3\eta+c_4)-c_1\tanh(c_1\eta+c_2)]^2}{\frac{1}{2}c_3\tanh(c_3\eta+c_4)-2c_1\tanh(c_1\eta+c_2)}\end{aligned} \tag{5.27}$$

The relative anisotropy:

$$\frac{\sigma^2}{\rho}=\frac{2c_3}{3c_1^4}\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{sech}^4(c_1\eta+c_2)}$$
$$\times[c_3\tanh(c_3\eta+c_4)-c_1\tanh(c_1\eta+c_2)]^2$$
$$\times\left[-\frac{3}{4\chi}\beta^2+\frac{c_3}{\chi c_1^4}\left(c_1^2-\frac{c_3^2}{4}\right)\frac{\operatorname{sech}(c_3\eta+c_4)}{\operatorname{sech}^4(c_1\eta+c_2)}\right]^{-1} \tag{5.28}$$

APPENDIX

The general metric for Bianchi type II, VIII, and IX models is

$$dS^2=dt^2-S^2\,dx^2-R^2\,dy^2-(R^2f^2+S^2h^2)\,dz^2+2S^2h\,dx\,dz \tag{A.1}$$

where

$$R=R(t),\qquad S=S(t),\qquad h=h(y),\qquad f=f(y)$$

$$f(y)=\begin{vmatrix}\sin y\\ y\\ \sinh y\end{vmatrix},\qquad h(y)=\begin{vmatrix}\cos y\\ -\frac{1}{2}y^2\\ -\cosh y\end{vmatrix}\qquad \text{for}\quad \delta=\begin{vmatrix}+1\\ 0\\ -1\end{vmatrix}\begin{matrix}\text{IX}\\ \text{II}\\ \text{VIII}\end{matrix} \tag{A.2}$$

The proper spatial volume is $V^3=\sqrt{-g}=SR^2f(y)$.
The field equations (2.1)–(2.4) for the metric (A.1) are

$$2\frac{\ddot{R}}{R}+\left(\frac{\dot{R}}{R}\right)^2-\frac{1}{R^2}\left(\frac{f_{22}}{f}\right)-\frac{3}{4}\frac{h_2^2S^2}{f^2R^4}-\left(\frac{h_{22}}{h}-\frac{h_2f_2}{hf}\right)\frac{h^2S^2}{2f^2R^4}$$
$$=-\chi p-\tfrac{3}{4}\beta^2 \tag{A.3}$$

$$\frac{\ddot{R}}{R}+\frac{\ddot{S}}{S}+\frac{\dot{R}\dot{S}}{RS}+\frac{1}{4}\frac{h_2^2S^2}{f^2R^4}$$
$$=-\chi p-\tfrac{3}{4}\beta^2 \tag{A.4}$$

$$\frac{\ddot{R}}{R}+\frac{\ddot{S}}{S}+\frac{\dot{R}\dot{S}}{RS}+\frac{1}{4}\frac{h_2^2S^2}{f^2R^4}-\frac{S^2h^2}{2f^2R^4}\left(-\frac{h_{22}}{h}+\frac{h_2f_2}{hf}\right)$$
$$=-\chi p-\tfrac{3}{4}\beta^2 \tag{A.5}$$

$$\left(\frac{\dot{R}}{R}\right)^2+2\frac{\dot{R}\dot{S}}{RS}-\frac{1}{R^2}\left(\frac{f_{22}}{f}\right)-\frac{1}{4}\frac{h_2^2S^2}{f^2R^4}$$
$$=\chi\rho+\tfrac{3}{4}\beta^2 \tag{A.6}$$

$$h\left[\frac{\ddot{R}}{R}+\left(\frac{\dot{R}}{R}\right)^2-\frac{\ddot{S}}{S}-\frac{\dot{R}\dot{S}}{RS}\right]$$
$$+\frac{1}{2}\frac{h}{R^2}\left(\frac{h_{22}}{h}-\frac{h_2 f_2}{hf}-2\frac{f_{22}}{f}\right)$$
$$-\frac{1}{2}\frac{h_{22}h^2S^2}{f^2R^4}-\frac{S^2h^2h_2^2}{f^2R^4}\left(1-\frac{1}{2}\frac{f_2}{f}\frac{h}{h_2}\right)=0 \qquad \text{(A.7)}$$

An overdot denotes differentiation with respect to t, and h_2, f_{22}, etc., stand for derivatives with respect to y.

The energy conservation equation is

$$\chi\dot{\rho}+\frac{3}{2}\beta\dot{\beta}+\left[\chi(\rho+p)+\frac{3}{2}\beta^2\right]\left(\frac{\dot{S}}{S}+2\frac{\dot{R}}{R}\right)=0 \qquad \text{(A.8)}$$

The Ricci scalar is

$$R=2\left[\frac{\ddot{S}}{S}+2\frac{\ddot{R}}{R}+\left(\frac{\dot{R}}{R}\right)^2+2\frac{\dot{R}\dot{S}}{RS}-\frac{1}{R^2}\left(\frac{f_{22}}{f}\right)-\frac{1}{4}\frac{h_2^2S^2}{f^2R^4}\right] \qquad \text{(A.9)}$$

The expansion (θ) and shear (σ^2) scalars are

$$\theta=\left(\frac{\dot{S}}{S}+2\frac{\dot{R}}{R}\right) \quad \text{and} \quad \sigma^2=\frac{2}{3}\left(\frac{\dot{R}}{R}-\frac{\dot{S}}{S}\right)^2 \qquad \text{(A.10)}$$

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